Topic: Listing the solutions in the interval

Question: Find all the solutions of the equation.

$$\cos\left(\theta + \frac{3\pi}{2}\right) = \frac{\sqrt{3}}{2}$$

Answer choices:

A All angles θ that satisfy either $\theta = \left\{\frac{2\pi}{3} + 2n\pi\right\}$ or $\theta = \left\{\frac{4\pi}{3} + 2n\pi\right\}$

B All angles θ that satisfy either $\theta = \left\{\frac{\pi}{6} + 2n\pi\right\}$ or $\theta = \left\{\frac{5\pi}{6} + 2n\pi\right\}$

C All angles θ that satisfy either $\theta = \left\{\frac{\pi}{4} + 2n\pi\right\}$ or $\theta = \left\{\frac{3\pi}{4} + 2n\pi\right\}$

D All angles θ that satisfy either $\theta = \left\{\frac{\pi}{3} + 2n\pi\right\}$ or $\theta = \left\{\frac{2\pi}{3} + 2n\pi\right\}$

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Solution: D

By the sum identity for cosine,

$$\cos\left(\theta+\frac{3\pi}{2}\right)=\cos\theta\left[\cos\left(\frac{3\pi}{2}\right)\right]-\sin\theta\left[\sin\left(\frac{3\pi}{2}\right)\right]$$

Since $\cos(3\pi/2)=0$ and $\sin(3\pi/2)=-1$, we have

$$\cos\left(\theta+\frac{3\pi}{2}\right)=(\cos\theta)(0)-(\sin\theta)(-1)=\sin\theta$$

Thus we need to find all the solutions of the equation

$$\sin\theta=\frac{\sqrt{3}}{2}$$

Well, we know that the acute angle that has a sine of $\sqrt{3}/2$ is $\pi/3$. The only other quadrant in which the sine is positive is the second quadrant. Therefore, every angle in the second quadrant that has a sine of $\sqrt{3}/2$ is an angle whose reference angle is $\pi/3$. One such angle is

$$\pi-\frac{\pi}{3}=\frac{3(\pi)-1(\pi)}{3}=\frac{3\pi-\pi}{3}=\frac{2\pi}{3}$$

From these results, we find that the solutions of the equation

$$\cos\left(\theta+\frac{3\pi}{2}\right)=\frac{\sqrt{3}}{2}$$

are all the angles θ for which there is some integer n that satisfies one of the following two equations:

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$$\theta = \frac{\pi}{3} + 2n\pi$$

$$\theta = \frac{2\pi}{3} + 2n\pi$$

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Topic: Listing the solutions in the interval

Question: What are the solutions of the equation that lie in the interval?

$$\csc^2\theta + \csc\theta = 2 \text{ on } [0,2\pi)$$

Answer choices:

A $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{\pi}{2}$

B $\theta = \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{3\pi}{2}$

C $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

D $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}$

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Solution: A

Note that

$$\csc^2\theta + \csc\theta = 2 \iff \csc^2\theta + \csc\theta - 2 = 0$$

Therefore, the solutions of the equation $\csc^2\theta + \csc\theta = 2$ are the same as the solutions of the equation

$$\csc^2\theta + \csc\theta - 2 = 0$$

Now we'll think of $\csc\theta$ as a variable (say v), so solving the equation

$$\csc^2\theta + \csc\theta - 2 = 0$$

is tantamount to solving the equation

$$v^2 + v - 2 = 0$$

Well, we can factor $v^2 + v - 2$ as $(v + 2)(v - 1)$, so this gives us the equation

$$(\csc\theta + 2)(\csc\theta - 1) = 0$$

Thus we see that the solution set of the equation

$$\csc^2\theta + \csc\theta = 2$$

consists of the union of the solution set of the equation

$$\csc\theta + 2 = 0$$

and the solution set of the equation

$$\csc\theta - 1 = 0$$

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Now

$$\csc\theta + 2 = 0 \iff \csc\theta = -2$$

and

$$\csc\theta - 1 = 0 \iff \csc\theta = 1$$

Using the reciprocal identity

$$\csc\theta = \frac{1}{\sin\theta}$$

we find that we need to solve the following two equations:

$$\frac{1}{\sin\theta} = -2$$

$$\frac{1}{\sin\theta} = 1$$

Let's solve each of these equations separately, starting with

$$\frac{1}{\sin\theta} = -2$$

Multiplying both sides of this equation by $\sin\theta$, we get

$$1 = -2\sin\theta$$

Dividing both sides by -2:

$$-\frac{1}{2} = \sin\theta$$

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The only quadrants in which there are angles whose sine is negative are the third and fourth quadrants. We know that the acute angle that has a sine of $+1/2$ is $\pi/6$. Thus one angle in the third quadrant whose reference angle is $\pi/6$ is

$$\frac{\pi}{6} + \pi = \frac{1(\pi) + 6(\pi)}{6} = \frac{7\pi}{6}$$

Also, one angle in the fourth quadrant whose reference angle is $\pi/6$ is

$$2\pi - \frac{\pi}{6} = \frac{6(2\pi) - 1(\pi)}{6} = \frac{12\pi - \pi}{6} = \frac{11\pi}{6}$$

Thus the solutions of the equation $\sin\theta = -1/2$ are the angles θ for which there is some integer n that satisfies one of the following two equations:

$$\theta = \frac{7\pi}{6} + 2n\pi$$

$$\theta = \frac{11\pi}{6} + 2n\pi$$

The only solutions that lie in the interval $[0,2\pi)$ are $7\pi/6$ and $11\pi/6$.

Now we'll find the solutions of the equation

$$\frac{1}{\sin\theta} = 1$$

Well, multiplying both sides of this equation by $\sin\theta$ gives

$$1 = \sin\theta$$

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As we know, the angles θ with $\sin\theta = 1$ are those for which there is some integer n such that

$$\theta = \frac{\pi}{2} + 2n\pi$$

The only solution of the equation $1 = \sin\theta$ that lies in the interval $[0,2\pi)$ is $\theta = \pi/2$.

Combining results, we find that the solutions of the equation

$$\csc^2\theta + \csc\theta = 2$$

that lie in the interval $[0,2\pi)$ are $7\pi/6$, $11\pi/6$, and $\pi/2$.

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Topic: Listing the solutions in the interval

Question: What are the solutions of the equation that lie in the interval?

$$\tan(4\theta + \pi) = -1 \text{ on } [0,2\pi)$$

Answer choices:

A $\theta = \frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}, \frac{17\pi}{16}, \frac{21\pi}{16}, \frac{25\pi}{16}, \frac{29\pi}{16}$

B $\theta = \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

C $\theta = \frac{3\pi}{16}, \frac{7\pi}{16}, \frac{11\pi}{16}, \frac{15\pi}{16}, \frac{19\pi}{16}, \frac{23\pi}{16}, \frac{27\pi}{16}, \frac{31\pi}{16}$

D $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$

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Solution: C

By the sum identity for tangent,

$$\tan(4\theta + \pi) = \frac{\tan(4\theta) + \tan(\pi)}{1 - \tan(4\theta)\tan(\pi)}$$

Since $\tan \pi = 0$, we have

$$\tan(4\theta + \pi) = \frac{\tan(4\theta) + 0}{1 - \tan(4\theta)(0)} = \frac{\tan(4\theta)}{1} = \tan(4\theta)$$

Thus we need to solve the equation

$$\tan(4\theta) = -1$$

The only quadrants in which the tangent is negative are the second and fourth quadrants. The acute angle whose tangent is $+1$ is $\pi/4$, so any angle 4θ whose tangent is equal to -1 has reference angle $\pi/4$ and is in the second or fourth quadrant. One such angle in the second quadrant is

$$4\theta = \pi - \frac{\pi}{4} = \frac{4(\pi) - 1(\pi)}{4} = \frac{4\pi - \pi}{4} = \frac{3\pi}{4}$$

Similarly, one angle 4θ in the fourth quadrant whose tangent is equal to -1 is

$$4\theta = 2\pi - \frac{\pi}{4} = \frac{4(2\pi) - 1(\pi)}{4} = \frac{8\pi - \pi}{4} = \frac{7\pi}{4}$$

Thus every angle θ that is a solution of the equation $\tan(4\theta + \pi) = -1$ has the property that there is some integer n that satisfies one of the following two equations:

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$$4\theta = \frac{3\pi}{4} + 2n\pi$$

$$4\theta = \frac{7\pi}{4} + 2n\pi$$

Dividing both sides of each of these two equations by 4, we find that $\tan(4\theta + \pi) = -1$ if and only if there is some integer n that satisfies one of the following two equations:

$$\theta = \frac{3}{16} + \frac{n\pi}{2}$$

$$\theta = \frac{7\pi}{16} + \frac{n\pi}{2}$$

Note that when $n = 0$, the angles that satisfy these two equations are $3\pi/16$ and $7\pi/16$, respectively. Now

$$0 < \frac{3\pi}{16} < \frac{32\pi}{16} = 2\pi$$

and

$$0 < \frac{7\pi}{16} < \frac{32\pi}{16} = 2\pi$$

Therefore, both of these angles are in the interval $[0,2\pi)$.

Now let $n = 1$. Then we get the angles

$$\frac{3\pi}{16} + \frac{1(\pi)}{2} = \frac{1(3\pi) + 8(\pi)}{16} = \frac{3\pi + 8\pi}{16} = \frac{11\pi}{16}$$

and

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$$\frac{7\pi}{16}+\frac{1(\pi)}{2}=\frac{1(7\pi)+8(\pi)}{16}=\frac{7\pi+8\pi}{16}=\frac{15\pi}{16}$$

respectively. Note that

$$0<\frac{11\pi}{16}<\frac{32\pi}{16}=2\pi$$

and

$$0<\frac{15\pi}{16}<\frac{32\pi}{16}=2\pi$$

Thus the angles $11\pi/16$ and $15\pi/16$ are also solutions that lie in the interval $[0,2\pi)$.

How about $n=2$? Well,

$$\frac{3\pi}{16}+\frac{2(\pi)}{2}=\frac{1(3\pi)+8(2\pi)}{16}=\frac{3\pi+16\pi}{16}=\frac{19\pi}{16}$$

and

$$\frac{7\pi}{16}+\frac{2(\pi)}{2}=\frac{1(7\pi)+8(2\pi)}{16}=\frac{7\pi+16\pi}{16}=\frac{23\pi}{16}$$

respectively. Note that

$$0<\frac{19\pi}{16}<\frac{32\pi}{16}=2\pi$$

and

$$0<\frac{23\pi}{16}<\frac{32\pi}{16}=2\pi$$

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Thus both $19\pi/16$ and $23\pi/16$ are solutions that lie in the interval $[0,2\pi)$.

Dare we even think that the angles with $n = 3$ will lie in that interval? Well, here goes:

$$\frac{3\pi}{16} + \frac{3(\pi)}{2} = \frac{1(3\pi) + 8(3\pi)}{16} = \frac{3\pi + 24\pi}{16} = \frac{27\pi}{16}$$

and

$$\frac{7\pi}{16} + \frac{3(\pi)}{2} = \frac{1(7\pi) + 8(3\pi)}{16} = \frac{7\pi + 24\pi}{16} = \frac{31\pi}{16}$$

respectively. Now

$$0 < \frac{27\pi}{16} < \frac{32\pi}{16} = 2\pi$$

and

$$0 < \frac{31\pi}{16} < \frac{32\pi}{16} = 2\pi$$

We have found that both $27\pi/16$ and $31\pi/16$ are also in the interval $[0,2\pi)$.

Let's go on to $n = 4$:

$$\frac{3\pi}{16} + \frac{4(\pi)}{2} = \frac{1(3\pi) + 8(4\pi)}{16} = \frac{3\pi + 32\pi}{16} = \frac{35\pi}{16}$$

and

$$\frac{7\pi}{16} + \frac{4(\pi)}{2} = \frac{1(7\pi) + 8(4\pi)}{16} = \frac{7\pi + 32\pi}{16} = \frac{39\pi}{16}$$

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respectively. Note that

$$\frac{35\pi}{16} > \frac{32\pi}{16} = 2\pi$$

and

$$\frac{39\pi}{16} > \frac{32\pi}{16} = 2\pi$$

Thus neither of the solutions for $n = 4$ lies in the interval $[0,2\pi)$.

How about negative integers n? First, we'll check $n = -1$:

$$\frac{3\pi}{16} + \frac{(-\pi)}{2} = \frac{1(3\pi) + 8(-\pi)}{16} = \frac{3\pi - 8\pi}{16} = -\frac{5\pi}{16} < 0$$

and

$$\frac{7\pi}{16} + \frac{(-\pi)}{2} = \frac{1(7\pi) + 8(-\pi)}{16} = \frac{7\pi - 8\pi}{16} = -\frac{\pi}{16} < 0$$

respectively. Since both of these angles are less than 0, they are not in the interval $[0,2\pi)$. Also, it is easy to see that the same is true for all $n \leq -2$, since those angles are even "more negative" than the ones for $n = -1$.

Collecting all our results, we find that the solutions of the equation

$$\tan(4\theta + \pi) = -1$$

that lie within the interval $[0,2\pi)$ are

$$\frac{3\pi}{16}, \frac{7\pi}{16}, \frac{11\pi}{16}, \frac{15\pi}{16}, \frac{19\pi}{16}, \frac{23\pi}{16}, \frac{27\pi}{16}, \frac{31\pi}{16}$$

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